JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 22, No. 1, January–February 1999

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Stability Analysis of Baumgarte Constraint Stabilization Technique in Multibody Dynamic Systems

J. C. Chiou,* J. Y. Yang,† and S. D. Wu†
National Chiao-Tung University,
Hsinchu 300, Taiwan, Republic of China

Introduction

COMPUTER simulation of multibody dynamic (MBD) systems is important for its capability to design and to analyze engineering problems such as robot arm manipulators and ground vehicles. In general, the dynamic equations of MBD systems can be derived and expressed in a set of differential algebraic equations (DAE). In the past two decades, computational methods such as the Baumgarte constraint stabilization technique,[1,2] the adaptive constraint violation stabilization method,[3] the coordinate partitioning technique,[4] the differential algebraic approach,[5,6] the penalty staggered stabilized procedure,[7] the gradient feedback method,[8] and the variable structure system control technique[9] have been proposed to solve DAE.

Of the techniques just cited, the Baumgarte technique is considered to be the easiest technique to stabilize the constraint violations of DAE during the process of numerical time integration. The Baumgarte technique simply modified the original constraint equations to form a set of relaxation differential constraint equations to stabilize the constraint violations. However, the success of applying this technique is heavily dependent on the choice of the stabilization parameters α and β. In this Note a guideline based on the stability limits of the employed numerical algorithms and the chosen integration step size is proposed to systematically determine these stabilization parameters.

Dynamic Equations of Multibody Dynamic Systems with Holonomic Constraints

The discrete equations of motion for MBD systems with holonomic constraints can be derived and expressed in the following DAE form[7]:

$$\boldsymbol{M}\ddot{q} + \boldsymbol{B}^T\lambda = F(q, \dot{q}, t) \tag{1a}$$

$$\Phi(q) = 0 \tag{1b}$$

where $\boldsymbol{M} \in R^{n\times n}$ denotes the mass matrix of the system, $q \in R^n$ are the generalized coordinate components, $\Phi \in R^m$ are the constraint conditions, $\boldsymbol{B} \in R^{m\times n}$ are the Jacobian matrix of the constraint equations, $\lambda \in R^m$ is the Lagrange multiplier, and $F(q, \dot{q}, t) \in R^m$ are the applied forces. Because the solution procedures of Eqs. (1a) and (1b) suffer various drawbacks, such as constraint violation, and are numerically stiff in the computer implementation,[1,2] researchers such as Baumgarte have been motivated to look for alternative solution procedures that overcome these difficulties. In the Baumgarte technique, one replaces Eq. (1b) by

$$\ddot{\Phi} + 2\alpha\dot{\Phi} + \beta^2\Phi = 0 \tag{2}$$

where α and β are both positive real numbers. Combining Eqs. (1a) and (2) with $\dot{\Phi} = B\dot{q}$ and $\ddot{\Phi} = B\ddot{q} + \dot{B}\dot{q}$, we obtain the following system equations:

$$\begin{bmatrix} M & B^T \\ B & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F(q, \dot{q}, t) \\ -\dot{B}\dot{q} - 2\alpha\dot{\Phi} - \beta^2\Phi \end{Bmatrix} \tag{3}$$

Equation (3) has been successfully applied to many MBD systems. Hahn and Simeon[10] indicated that despite the practical success of applying Eq. (3) to a number of MBD systems, the Baumgarte technique is "not yet fully explored" from the system theoretical point of view and to some extent is still "unattractive for general-purpose use." Mainly, the Baumgarte technique encounters the following difficulties: 1) the dynamic cross coupling among the subsystems given in Eq. (3) has not yet been investigated systematically, and 2) the choice of the stabilization parameters is arbitrary. To deal with the first difficulty, a constraint separation principle proposed by Hahn and Simeon[10] is summarized as follows:

The characteristic polynomial $\psi(s; \alpha, \beta)$ of system equation (3) linearized about an equilibrium solution can be written as a product of two polynomials

$$\psi(s; \alpha, \beta) = \psi_B(s; \alpha, \beta) \cdot \psi_P(s) \tag{4}$$

The first polynomial factor, $\psi_B(s; \alpha, \beta)$, is the characteristic polynomial of the linearized constraint stabilization relation associated with Eq. (2). The second polynomial factor, $\psi_P(s)$, is the characteristic polynomial of the linearized model equations (1a) and (1b) in conjunction with the elimination of λ. The roots of $\psi_P(s)$ are identical to the physical eigenvalues of the linearized constrained dynamic system equations and are independent of α and β.

The constraint separation principle of Hahn and Simeon is a local theorem that justifies the reliability of the Baumgarte technique. Recently, Chiou and Wu[9] resolve this problem from the differential geometric point of view. They observed that the zero dynamics of DAE and the ordinary differential equation of the MBD systems are equivalent. The work of Hahn and Simeon and Chiou and Wu indicated that the Baumgarte technique does not affect the dynamics of the original MBD system around the equilibrium solutions. In other words, the numerical stability of the original MBD systems and Eq. (2) can be discussed separately if the constraint equation (1b) is not violated. With the success of overcoming the first difficulty, a method based on the stability limits of the employed numerical algorithms and the chosen integration step size is proposed to overcome the second difficulty.

Stability Limits of α and β

Note that Eq. (2) proposed by the Baumgarte technique is a second-order linear differential equation of constraint equation Φ. Chang and Nikravesh[3] suggested that the Baumgarte technique is at its best by choosing α and β at a critical damping condition, i.e., $\alpha = \beta$. With this condition, the characteristic equation of Eq. (2) yields

$$D^2 + 2\alpha D + \alpha^2 = 0 \tag{5}$$

Received May 18, 1998; revision received Aug. 30, 1998; accepted for publication Aug. 30, 1998.

*Associate Professor, Department of Electrical and Control Engineering, 1001 Ta Hsueh Road. E-mail: chiou@cc.nctu.edu.tw. Member AIAA.

†Research Assistant, Department of Electrical and Control Engineering, 1001 Ta Hsueh Road.

where D is the characteristic root. The roots of the preceding characteristic equation are

$$D = -\alpha, -\alpha \tag{6}$$

Because the products of the integration step size and the roots must lie within the stability limit of the employed numerical integration algorithm to ensure its stability, the following relationship is established:

$$\alpha \cdot h \leq SL \tag{7}$$

where h is the integration step size and SL is the stability limit of the employed integration algorithm. From Eq. (5) we observe that the magnitude of α will greatly affect the position and velocity constraint errors during the process of time integration. In other words, a large α will stabilize the constraint violation more efficiently than a small α. However, the MBD systems will become numerically unstable if inappropriately large α is chosen. To obtain a proper α, the equality condition of Eq. (7) is imposed here to give the following stabilization parameter:

$$\alpha = SL/h \equiv Sh \tag{8}$$

Two important features in determining the value of α are observed from Eq. (8): the stability limit for the employed numerical integrator and the chosen integration step size. However, if the stabilization parameter α, obtained from Eq. (8), is directly applied to Eq. (3), constraint violations are still observed from our numerical experiments. This is because Eq. (8) is obtained from the constraint separation principle, which assumes that the constraint equations are always satisfied. Because numerical integration algorithms are an approximation, constraint errors will occur during the process of time integration. Thus, we cannot exactly decouple Eq. (3) from the constraint separation principle point of view. To overcome this difficulty, the concept of a safety factor is employed here to take into account the coupling and to ensure that the constraint violations remain within a certain truncation error. The optimal α is obtained by multiplying Eq. (8) by an SF as

$$\alpha_{\text{opt}} = SF^{*}Sh \tag{9}$$

where SF is the safety factor that needs to be decided from numerical experiments. A guideline based on the preceding development is obtained and summarized as follows:

1) Employ an integration algorithm and obtain its corresponding stability limit SL.

2) Choose the integration step size h, and set the safety factor, $SF = 1$.

3) Give initial conditions of Eq. (3).

4) Calculate the stabilization parameter from Eq. (9).

5) Use the employed integration algorithm to integrate Eq. (3).

6) Check the constraint errors. If the constraint errors are increased over the simulation time period, then set $SF = SF - 0.1$, and go to step 3; otherwise, stop.

A numerical example is conducted in the next section to demonstrate the effectiveness of the proposed guideline.

Numerical Example—Crank-Mechanism Problem

The equations of motion of the crank mechanism are characterized by the following matrices and constraints:

$$M = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & m & \\ & & & m \end{bmatrix} \tag{10}$$

$$\Phi = \begin{bmatrix} r\cos\theta - (x - l_1\cos\phi) \\ r\sin\theta - (y - l_1\sin\phi) \\ (l - l_1)\sin\phi + y \end{bmatrix} = 0$$

$$q = [\theta \quad \phi \quad x \quad y]^T, \qquad F = \{0 \quad 0 \quad 0 \quad -mg\}^T \tag{11}$$

Table 1 Stability limits of Adams–Bashforth integrator

Order k	2	3	4	5
SL	1.0	0.5454	0.3	0.1633

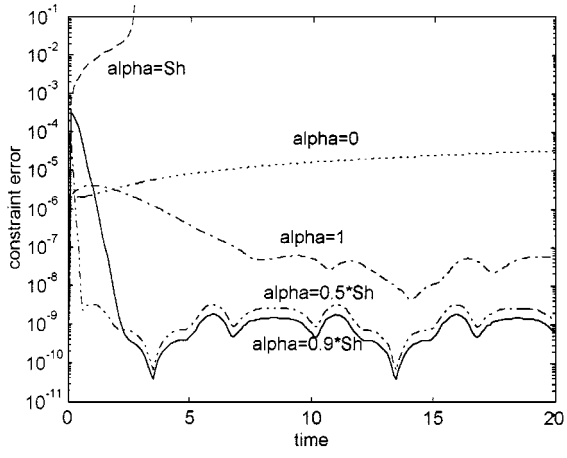

Fig. 1 2-norm of constraint errors.

where $J_1 = 0.045$, $J_2 = 33/4800$, $m = 1$, $r = 0.3$, $l = 0.5$, and $l_1 = 0.3$. The initial conditions for the example are

$$\begin{aligned} \theta(0) &= 0.9851, & \phi(0) &= -0.5236 \\ x(0) &= 0.4256, & y(0) &= 0.1 \\ \dot{\theta}(0) &= \pi/5, & \dot{\phi}(0) &= -0.240628 \\ \dot{x}(0) &= -0.193174, & \dot{y}(0) &= 0.041678 \end{aligned} \tag{12}$$

To proceed with the proposed guideline, the family of Adams–Bashforth integrators is chosen for its explicit nature. Table 1 presents the stability limits of the Adams–Bashforth integrator of order k from 2 to 5.

In the current simulation, the Adams–Bashforth third-order (AB-3) integration algorithm is used with step size $h = 0.01$ s. According to Eq. (8), the resulting stabilization parameter $Sh = 54.54$. Note that the starting procedure of AB-3 is resolved by using the Runge–Kutta fourth-order integration with an adaptive step size. Figure 1 shows the 2-norm of constraint errors when $\alpha = 0$, 1, Sh, $0.5^{*}Sh$, and $0.9^{*}Sh$, respectively. From Fig. 1 we observe that a larger α yields a smaller constraint error where the robustness against numerical truncation error is lost. For the case of $\alpha \geq Sh$, regardless of the step size, the MBD system becomes numerically unstable. To obtain a balance between the accuracy and the robustness in simulating MBD systems, we examine the proposed guideline with a safety factor equal to 0.5 and 0.9. The simulation results indicate that $\alpha = 0.5^{*}Sh$ and $0.9^{*}Sh$ yield the most satisfactory results in constraint errors. To complete the current study, Adam–Bashforth integrators with different orders (second, fourth, and fifth) and step sizes were also conducted. Similar results are obtained according to the proposed guideline.

Conclusion

A guideline is developed to determine the stabilization parameters of the Baumgarte technique. The guideline, based on the stability limits of the employed numerical algorithms and the chosen integration step size, is used to systematically determine the stabilization parameter. Numerical results indicated that the current guideline yields a more accurate and robust solution than arbitrarily determined stabilization parameters in simulating MBD systems.

Acknowledgment

The present work was supported by the National Science Council under Grant NSC 82-0113-E009-412-T.

References

[1]Baumgarte, J. W., "Stabilization of Constraints and Integrals of Motion in Dynamical System," *Computational Method in Applied Mechanics and Engineering*, Vol. 1, No. 4, 1972, pp. 1–16.

[2]Baumgarte, J. W., "A New Method of Stabilization for Holonomic Constraints," *Journal of Applied Mechanics*, Vol. 50, Dec., 1983, pp. 869, 870.

[3]Chang, C. O., and Nikravesh, P. E., "An Adaptive Constraint Violation Stabilization Analysis of Mechanical Systems," *Transactions of the American Society of Mechanical Engineers*, Vol. 107, Dec. 1985, pp. 488–492.

[4]Wehage, R. A., and Haug, E. J., "Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic Systems," *Journal of Mechanical Design*, Vol. 104, Jan. 1982, pp. 247–255.

[5]Gear, C. W., "Simultaneous Numerical Solution of Differential/Algebraic Equations," *IEEE Transactions on Circuit Theory*, CT-18, Jan. 1971, pp. 89–95.

[6]Pezold, L., "Differential/Algebraic Equations Are Not ODEs," *SIAM Journal on Scientific and Statistical Computing*, Vol. 3, N0. 3, 1982, pp. 367–384.

[7]Park, K. C., and Chiou, J. C., "Stabilization of Computational Procedures for Constrained Dynamical Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 4, 1988, pp. 365–370.

[8]Yoon, S., Howe, R. M., and Greenwood, D. T., "Constraint Violation Stabilization Using Gradient Feedback in Constrained Dynamics Simulation," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 6, 1992, pp. 1467–1474.

[9]Chiou, J. C., and Wu, S. D., "Constraint Violation Stabilization Using Input-Output Feedback Linearization in Multibody Dynamic Analysis," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 2, 1998, pp. 222–228.

[10]Hahn, H., and Simeon, B., "Separation Principle of Mechanical System Models Including Stabilized Constraint Relations," *Applied Mechanics*, Vol. 64, No. 3, 1994, pp. 147–153.

Expected Residual Vibration of Traditional and Hybrid Input Shaping Designs

Lucy Y. Pao* and Mark A. Lau†
University of Colorado, Boulder, Colorado 80309-0425

Introduction

IN the modeling of flexible structures, there is generally uncertainty in both the frequencies and damping constants; thus, it is important that the control methods for such systems be robust to these uncertainties. Input shaping has been successfully applied for controlling flexible structures while being insensitive to modeling errors.[1,2] However, previous studies have primarily used a measure of insensitivity that does not incorporate information about the probability distribution of parameter variations. The probability distribution of parameters can often be approximated in particular applications. For instance, in mass manufactured disk-drive systems, variations in the read/write arm mass and length can be mapped to determine the expected variations in the natural frequency. Similarly, operating temperature changes can affect the viscosity of coatings and alter the damping coefficients of the system. In such applications, data can be taken to estimate the distribution of parameters.

We investigate a recently proposed input shaping method[3] that suggests a measure of performance in terms of the expected level of residual vibration and easily incorporates the probability distribution of parameter variations. We also propose a very straightforward and computationally efficient technique to derive shapers that yield near-optimal performance. An analysis is presented for several types of optimal and near-optimal input shapers, and some interesting trends and qualitative behaviors of the shapers are shown.

Received Feb. 17, 1998; revision received Sept. 9, 1998; accepted for publication Sept. 11, 1998.

*Assistant Professor, Electrical and Computer Engineering Department. E-mail: pao@colorado.edu. Member AIAA.

†Research Assistant, Electrical and Computer Engineering Department. E-mail: mark.lau@colorado.edu.

Expected Residual Vibration of Shaper Designs

For simplicity of exposition, we limit the discussion to one-bending-mode flexible structures. The residual vibration resulting from a sequence of impulses is[1]

$$V(\omega, \zeta, t_v) = e^{-\zeta\omega t_v} \left\{ \left[\sum_{i=1}^{m} A_i e^{\zeta\omega t_i} \cos(\omega_d t_i) \right]^2 + \left[\sum_{i=1}^{m} A_i e^{\zeta\omega t_i} \sin(\omega_d t_i) \right]^2 \right\}^{\frac{1}{2}} \tag{1}$$

where $\omega_d = \omega\sqrt{(1-\zeta^2)}$; there are m impulses, $i = 1, \ldots, m$; A_i is the amplitude of the ith impulse; t_i is the time location of the ith impulse; and $t_v \geq t_m$ is the time (after the end of the shaped input command) at which the residual vibration is computed.

Constraints to account for actuator limits are also included. Requiring that the impulse amplitudes are positive and sum to unity

$$A_i \geq 0, \qquad \sum_{i=1}^{m} A_i = 1, \qquad i = 1, \ldots, m \tag{2}$$

guarantees that 1) the final set point of the system is the same for shaped commands as for unshaped commands and 2) the shaper can be used with any unshaped input without violating the actuator limits if the original unshaped command does not violate them.[1]

In practice, we may have some knowledge of the statistical nature of plant parameter variation, and it may be useful to incorporate this knowledge into a performance measure J that evaluates the expected level of residual vibration

$$J = \int_0^1 \int_0^\infty V(\omega, \zeta, t_v)\, f(\zeta, \omega)\, \mathrm{d}\omega\, \mathrm{d}\zeta \tag{3}$$

where f is a joint probability density function of the actual system frequency and damping. This performance measure has several advantages over the traditional insensitivity[1,2,4] measure: The robustness with respect to damping is taken into account, the frequency and damping intervals of concern are selectable, and these intervals can be weighted (with a probability density function).

A simpler performance index such as

$$J = \int_0^\infty V(\omega, \zeta, t_v)\, f(\omega)\, \mathrm{d}\omega \tag{4}$$

can be used if the damping variation is expected to be small. Similarly, the performance index

$$J = \int_0^1 V(\omega, \zeta, t_v)\, f(\zeta)\, \mathrm{d}\zeta \tag{5}$$

can be used if we are concerned primarily with damping uncertainty. In this Note, we consider one-mode systems, but additional modes can be easily incorporated by extending the optimization criterion to minimizing $\Sigma_i J_i$, $i = 1, \ldots, n$, where each J_i is the expected level of residual vibration (3), (4), or (5) due to each flexible mode.

We consider uniform and Gaussian distributions for parameter variation. For the performance indices (4) and (5) that only include variations in one parameter, with uniform distribution, the parameter p, which is either the natural frequency ω or the damping coefficient ζ, has the probability density function

$$f(p) = \begin{cases} \dfrac{1}{p_{\mathrm{hi}} - p_{\mathrm{lo}}}, & p \in [p_{\mathrm{lo}}, p_{\mathrm{hi}}] \\ 0, & \text{otherwise} \end{cases} \tag{6}$$